

考生注意: 1. 依次序作答, 只要標明題號, 不必抄題。

2. 答案必須寫在答案卷上, 否則不予計分, 並限以藍黑色筆作答。

3. 試題隨卷繳回。(餘詳詳閱試場規則)

The following problems may be answered in Chinese or English. You need to give all details in order to receive any point.

1. (25%)

(a) (10%) Prove that if  $A$  is an  $n \times n$  matrix, the following statements are equivalent, ie. all true or all false.

(i) The reduced row-echelon form of  $A$  is  $I_n$ .

(ii)  $A$  is expressible as a product of elementary matrices.

(b) (15%) Find the inverse of  $A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$  by using the row operations.

2. (10%) If  $Ax = b$  is a consistent system of  $m$  equations in  $n$  unknowns, under what conditions will it be true that the solution set is a subspace of  $R^n$ ? Explain.

3. (20%)

(a) (10%) If  $A$  &  $B$  are row equivalent matrices, then prove that a given set of column vectors of  $A$  forms a basis for the column space of  $A$  if and only if the corresponding column vectors of  $B$  form a basis for the column space of  $B$ .

(b) (10%) Find a basis for the column spaces of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

4. (15%) Let the vector space  $P_2$  have the inner product

$\langle p, q \rangle = \int_0^1 p(x)q(x)dx$ . Apply the Gram-Schmidt process to transform the standard basis  $S = \{1, x, x^2\}$  into an orthonormal basis.

5. (10%) Show that a least squares solution of  $Ax=b$  must satisfy the equality  $A^T Ax = A^T b$  (i.e.  $x = (A^T A)^{-1} A^T b$ ) [it is called the normal system associated with  $Ax=b$ ].

6. (20%) Let  $T: R^3 \rightarrow R^3$  be the linear operator given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 + 3x_3 \end{pmatrix}$$

(a) (5%) Find the standard matrix  $A$  for  $T$ .

(b) (7%) Find a basis for  $R^3$  relative to which the matrix for  $T$  is diagonal.

(c) (8%) Find  $A^5$ .