國立暨南國際大學九十二學年度碩士班研究生入學考試試題

第 2 節線性代數 適用:(資工所 414)

(本試題共 1 頁，第 1 頁)

考生注意: 1. 依次序作答，只要標明題號，不必抄題。
2. 答案必須寫在答案卷上，否則不予計分，並限以藍黑色筆作答。
3. 試題隨卷繳回。(餘請詳閱試場規則)

1. True or False ($10 \times 3\%$) (不須寫出計算過程，答錯每題倒扣3%)

(1) If $T: V \to W$ is a one-to-one linear transformation, then $T^{-1}: R(T) \to V$ is a linear transformation. ($R(T)$ means the rang of the linear transformation T.)

(2) If A is an $n \times n$ matrix. A has a QR-decomposition if and only if the orthogonal complement of the row space of A is R^n.

(3) There is an invertible 10×10 matrix that has 92 ones among its entries.

(4) If $A^2 = 0$ for a 10×10 matrix A, then the inequality $\text{rank}(A) \le 5$ must hold.

(5) If matrix A is similar to B and A is orthogonal, then B must be orthogonal as well.

(6) The singular values of any triangular matrix are the absolute values of its diagonal entries.

(7) If $\det(A) = \det(A^T)$, then matrix A must be symmetric.

(8) $\mathbf{g}_1 = 2 \sin x + \cos x$ and $\mathbf{g}_2 = 3 \cos x$ form a basis for the space spanned by $\mathbf{f}_1 = -2 \sin x + 3 \cos x$ and $\mathbf{f}_2 = 5 \sin x + \cos x$.

(9) If the rank of a square matrix A is 1, then all the nonzero vectors in the image of A are eigenvectors of A.

(10) If all the entries of matrices A and A^{-1} are integers, then the equation $\det(A) = \det(A^{-1})$ must hold.

(以下各題均須寫出計算過程方予計分)

2. (10%) (a) Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned} x_1 \quad\quad\quad - x_3 &= 6 \\ 2x_1 + x_2 - 2x_3 &= 0 \\ x_1 + x_2 \quad\quad\quad &= 9 \\ x_1 + x_2 - x_3 &= 3 \end{aligned}.$$

(5%) (b) Find the orthogonal projection of $\mathbf{b}$ on the column space of A.

3. (10%) Use diagonalization to compute A^{10} for $A = \begin{bmatrix} 4 & -3 \\ 5 & -4 \end{bmatrix}$.

4. For $b \in \mathbf{R}$, $b \neq 0$. For $n \geq 2$, consider the $n \times n$ matrix D_n given by $\begin{bmatrix} 0 & 0 & \cdots & 0 & b \\ 0 & 0 & \cdots & b & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & b & \cdots & 0 & 0 \\ b & 0 & \cdots & 0 & 0 \end{bmatrix}$.

(10%) (a) Find bases for the eigenspaces of D_n.

(10%) (b) Show that D_n has no LU-decomposition.

5. (10%) Prove: If Q is an orthogonal matrix, then each entry of Q is the same as its cofactor if $\det(Q) = 1$ and is the negative of its cofactor if $\det(Q) = -1$.

6. (15%) Prove: If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_3\}$ is a basis for V and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_3$ are vectors in W, not necessarily distinct, then there exists a linear transformation $T: V \to W$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$, $T(\mathbf{v}_2) = \mathbf{w}_2$, $\ldots$, $T(\mathbf{v}_n) = \mathbf{w}_n$.