國立暨南國際大學九十二學年度博士班研究生入學考試試題

411 離散數學（資工所適用）

（本試題共 / 頁，第 / 頁）

考生注意：1. 依次序作答，只要標明題號，不必抄題。
2. 答案必須寫在答案卷上，否則不予計分。
3. 試題隨卷繳回。

1.

(a) Ten students enter a locker room that contains 10 lockers. The first student opens all the lockers. The second student changes the status (from closed to open, or vice versa) of every other locker, starting with the second locker. The third student then changes the status of every third locker, starting at the third locker. In general, for $1 < k \leq 10$, the kth student changes the status of every kth locker, starting with the kth locker. After the tenth student has gone through the lockers, which lockers are left open? (10%)

(b) Answer (a) if 10 is replaced by $n \in \mathbf{Z}^+$, $n \geq 2$. And prove your answer is correct. (15%)

2. A **k-$L(2,1)$-labeling** of a graph G is a function $f: V(G) \to \{0, 1, 2, \ldots, k\}$ such that $\forall\, x, y \in V(G)$: (i) if $d(x, y) = 1$ then $|f(x) - f(y)| \geq 2$; (ii) if $d(x, y) = 2$ then $|f(x) - f(y)| \geq 1$, where $d(x, y)$ means the distance of x and y. Given a graph G, let $\lambda(G) = \min\{k : \text{there exists a } k\text{-}L(2,1)\text{-labeling of } G\}$. Prove the following statements.

(a) If G is a graph with maximum degree $\Delta(G)$, then $\lambda(G) \geq \Delta(G) + 1$. (10%)

(b) If T is a tree, then $\lambda(T) \leq \Delta(T) + 2$. (15%)

3.

(a) If 6 vertices are selected from a plane such that no three of them are on the same line. Draw a line between any two vertices and color each line with either blue or red. Prove that there exists a triangle whose three edges are of the same color. (5%)

(b) Repeat (a), but 6 is replaced by 17, 2 colors blue and red are replaced by 3 colors blue, red and yellow. (10%)

(c) Write a statement that generalizes the results of (a) and (b), and prove it. (10%)

4. A graph **pyramid** $PM(n)$ is defined recursively by:

$V(PM(1)) = \{(0; 1, 1)\} \cup \{(1; x, y) : 1 \leq x \leq 2, 1 \leq y \leq 2\}$;

$E(PM(1)) = \{(0; 1, 1)(1; x, y) : 1 \leq x \leq 2, 1 \leq y \leq 2\} \cup \{(1; 1, y)(1; 2, y), (1; x, 1)(1; x, 2) : 1 \leq x \leq 2, 1 \leq y \leq 2\}$.

And for $n \geq 2$,

$V(PM(n)) = V(PM(n-1)) \cup \{(n; x, y) : 1 \leq x \leq 2^n, 1 \leq y \leq 2^n\}$;

$E(PM(n)) = E(PM(n-1)) \cup \{(n; x, y)(n-1; \lceil x/2 \rceil, \lceil y/2 \rceil) : 1 \leq x \leq 2^n, 1 \leq y \leq 2^n\} \cup$
$\{(n; x, y)\,(n; x, y+1) : 1 \leq x \leq 2^n, 1 \leq y \leq 2^n - 1\} \cup \{(n; x, y)\,(n; x+1, y) : 1 \leq x \leq 2^n - 1, 1 \leq y \leq 2^n\}$.

(a) $|V(PM(n))| = ?$ (5%)

(b) Let $\varepsilon(n) = |E(PM(n))|$, find and solve a recursive relation for $\varepsilon(n)$, $n \geq 1$. (10%)

(c) Given a connected graph G, the **eccentricity** of a vertex v in $G = e(v) = \max\{d(v, x) : x \in V(G)\}$. The **diameter** of $G = \boldsymbol{d(G)} = \max\{e(v) : v \in V(G)\}$. The **radius** of $G = \boldsymbol{r(G)} = \min\{e(v) : v \in V(G)\}$. Find the values of $d(PM(n))$ and $r(PM(n))$. (10%)