國立暨南國際大學九十二學年度博士班研究生入學考試試題

422 工程數學〈電機所適用〉

（本試題共 2 頁，第 1 頁）

考生注意：1. 依次序作答，只要標明題號，不必抄題。
2. 答案必須寫在答案卷上，否則不予計分。
3. 試題隨卷繳回。

1. (20 pts) Consider the set of three finite-energy signals

$$\begin{aligned} s_1(t) &= 1, & 0 \le t \le 1 \\ s_2(t) &= \cos(2\pi t), & 0 \le t \le 1 \\ s_3(t) &= 1 - \sin^2(\pi t), & 0 \le t \le 1 \end{aligned}$$

Find an orthonormal basis for the signal space spanned by these three signals. In this question, the inner product of $x_1(t)$ and $x_2(t)$ over the time interval (t_1, t_2) is defined as

$$\langle x_1(t), x_2(t) \rangle = \int_{t_1}^{t_2} x_1(t) x_2^*(t) dt.$$

2. (a) (15 pts) Consider two independently and identically distributed (i.i.d.) random variables X and Y with probability density functions (pdf's)

$$f_X(x) = f_Y(x) = \frac{1}{a} \Pi\left(\frac{x}{a}\right),$$

where $\Pi(y)$ denotes the rectangular function which is 1 for $|y| < \frac{1}{2}$, and 0 otherwise. Compute the pdf of $Z = X + Y$ using characteristic functions.

(b) (15 pts) Please precisely describe what the central-limit theorem is.

3. (25 pts) Determine $y(t)$ by solving the following differential equations:

(a) $\dfrac{dy}{dt} = \sin t + \int_0^t y(t-\tau)\cos\tau \, d\tau$, and $y(0) = 1$.

(b) $\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} + x + y = 1 \\ 2\dfrac{dx}{dt} + \dfrac{dy}{dt} + x = 0 \end{cases}$, and $x(0) = 1$, $y(0) = 0$.

4. (25 pts) True or false? Please do justify your answers to get full scores. (Give a brief proof or explanation if it is true. Otherwise, give a counter example to show that it is false).

(a) A matrix A is called **idempotent** if $A^2 = A$. An invertible matrix A is called **orthogonal** if $A^{-1} = A^T$. Therefore, if the matrix A is both symmetric and idempotent, then the matrix $I - 2A$ must be orthogonal.

(b) In order to guarantee that all the solutions of the 2^{nd}-order linear differential equation $ay'' + by' + cy = 0$ (i.e., $a \neq 0$) are bounded on the interval $[0, \infty)$, all of the constant coefficients a, b and c must be nonnegative.

(c) Let A, B and C be three nonzero $n \times n$ matrices. If these three matrices satisfy the equality $AC = BC$, then it can be concluded that $A = B$.

(d) Let u and v be vectors in an inner product space R^n. The two vectors u and v are orthogonal if and only if $\| u + v \| = \| u - v \|$.

(e) A square matrix A is said to be **similar** to a matrix B if there is an invertible matrix P such that $A = P^{-1}BP$. If the square matrix A is similar to a matrix B, then A and B have the same determinants and eigenvalues.