考生注意: 1. 依次序作答，只要標明題號，不必抄題。
2. 答案必須寫在答案卷上，否則不予計分，並限以藍黑色筆作答。
3. 試題隨卷繳回。(餘請詳閱試場規則)

1. Let $\mathbf{v}_1 = (1, 2, 0, 3)$, $\mathbf{v}_2 = (4, 0, 5, 8)$, $\mathbf{v}_3 = (8, 1, 5, 6)$.
   (a) (5%) Show that the above vectors can form a basis for a three-dimensional subspace V of $\mathbf{R}^4$.
   (b) (10%) Use $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ to construct an orthonormal basis for the three-dimensional subspace V of $\mathbf{R}^4$.

2. (15%) Let $\mathbf{e}_i$ and $\mathbf{e}_j$ be two eigenvectors corresponding to two distinct eigenvalues λ_i and λ_j $(\lambda_i \neq \lambda_j)$ of a real symmetric matrix A $(A = A^T)$, respectively. Show that the two eigenvectors $\mathbf{e}_i$ and $\mathbf{e}_j$ are orthogonal to each other. That is, show that $\mathbf{e}_i^T \mathbf{e}_j = 0$.

3. (a) (10%) Show that if $x = e^t$ and $y = y(x)$, then
$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right).$$
   (b) (10%) Use the result of (a) to solve the following differential equation.
$$2x^3y''' - 4x^2y'' - 20xy' = 0;\ x > 0.$$

4. (15%) Let $\mathbf{\Phi}(t)$ be a fundamental matrix satisfying the system $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t)$, derive the general solution for the system $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{F}(t)$.

5. (15%) A random variable X undergoes the transformation $Y = 2X^2 + 3$. Find the density function of Y given that X has the density function $f_X(x)$.

6. Two random variables X and Y have means $\overline{X} = 1$ and $\overline{Y} = 2$, variances $\sigma_X^2 = 4$ and $\sigma_Y^2 = 1$, and a correlation coefficient $\rho_{XY} = 0.4$. New random variables W and V are defined by
$$V = -X + 2Y \qquad W = X - 3Y.$$
   (a) (8%) Find the means of V and W.
   (b) (8%) Find the variances of V and W.
   (c) (4%) Find the correlation coefficient ρ_{VW} of V and W.