Entrance Examination: Electromagnetic Theory

As shown in the attached figure, there is a parallel-plate waveguide filled with a non-homogeneous medium characterized by $\varepsilon(y)$, that is the medium is uniform along transverse direction (x-z plane), however, non-uniform along y direction. The structure is assumed to be infinite extent along x and z directions. The separation distance between the top and bottom metal plates are designated as h.
Since the structure is uniform along x-z plane, we have the two sets of characteristic solutions, orthogonal to each other, which are named as TE and TM modes (with respect to y direction). In this problem, because that they have the similar mathematical process and results, we only consider TM case for succinctness.

Part 1: Mathematical Background
For the TM^y mode, it means that the magnetic field $\underline{H}$ is on the x-z plane, which is perpendicular to the wave guiding direction, y-axis. Consequently, we may invoke the commonly used Debye potential, which is defined as:

$$\underline{H} = \nabla \times \underline{A} \tag{1}$$

where $$\underline{A} = \underline{y}_o \psi(x,y,z) \tag{2}$$

Problem 1 (10 points): Please substitute (2) into (1) to find out the magnetic field components (H_x, H_y and H_z) expressed in terms of the scalar potential $\psi(x,y,z)$.

Problem 2 (10 points): After obtaining the magnetic field components, you could substitute them into the Maxwell equations to get the electric field components (E_x, E_y and E_z). Notice that you have to express them in terms of the three magnetic field components.

Problem 3 (10 points): Since the structure is infinite extent along transverse direction, the scalar potential $\psi(x,y,z)$ was given below:

$$\psi(x,y,z) = \phi(y)\exp(-j\underline{k}_t \cdot \underline{\rho}) \tag{3}$$

where $\underline{k}_t = k_x \underline{x}_o + k_z \underline{z}_o$ and $\underline{\rho} = x\underline{x}_o + z\underline{z}_o$

Problem 4 (15 points): Substitution of (3) into the solutions to problem 1 and 2 and through some simple algebraic derivations, you could obtain a second order differential equation for the eigen-function $\phi(y)$ in the form listed below:

$$\frac{d}{dy}\left[p(y)\frac{d}{dy}\phi(y)\right] + q(y)\phi(y) = \lambda w(y)\phi(y) \tag{4}$$

Please compare your result with (4) and indicate the one-to-one corresponding scalar functions $p(y)$, $q(y)$, weighting function $w(y)$ and the corresponding eigenvalue λ, respectively.

Problem 5 (20 points): the equation (4) is named as *Sturm-Liouville differential equation*, under *some* conditions, the eigenvalues are real numbers and the associated eigenfunctions are orthogonal to each other. Please indicate *the specific condition* and prove the properties described previously.

Part 2: Physical Picture of waves guiding in such a waveguide
Problem 6 (20 points): For the limiting case, we assume that the dielectric medium

filled in the parallel-plate waveguide is air, that is, $\varepsilon(y)=1$. In this problem, we will study the guiding characteristic of waves in such a waveguide. First of all, the second order differential equation (4) can be transformed into the following equation:

$$\frac{d^2}{dy^2}\phi(y)+(k_o^2-k_t^2)\phi(y)=0 \tag{5}$$

Please write down the general solution to the above equation.

Since the two parallel plates are made up of perfect conductors, by the electromagnetic boundary condition the tangential electric field components should vanish on the surface of a perfect conductor. Therefore, subject to the boundary conditions, the general solution you obtained previously should converge to a sinusoidal function, please write down it. It is noted that the general solution must be *normalized* based on the following definition.

$$\int_0^h |\phi(y)|^2\,dy=1 \tag{6}$$

Besides, please write down the *dispersion relation of waves supported by such a waveguide* (the relation between propagation constant and frequency).

Problem 7(15 points): If you can successfully achieve the eigen-functions along y direction, you should be able to write down the electric and magnetic field components in an explicit form. In addition, you could observe that the waves are obliquely bouncing back and forth around the two parallel plates. For each mode, they have different propagation constants and the bouncing angles; however, the angle will vary in accordance with the variation of operation frequency. Please describe the physical phenomena of cutoff in such a waveguide.