科目：線性代數 適用：資工所

編號：414

考生注意：
1.依次序作答，只要標明題號，不必抄題。
2.答案必須寫在答案卷上，否則不予計分。
3.限用藍、黑色筆作答；試題須隨卷繳回。

The following problems may be answered in Chinese or English. You need to give all details in order to receive any point.

List of symbols

- $e^{ix} = cosx+isinx$
- $M_3(\mathfrak{R})$: the set of 3×3 matrices over $\mathfrak{R}$
- $tr(A)$: the trace of A
- P_n : all real polynomials of degree n or less

1. (10%) Let V be the vector space over the complex number of all functions for $\mathfrak{R}$ to $\mathbb{C}$, i.e. the space of all complex-valued functions on the real line. Let $f_1(x) = 1, f_2(x) = e^{ix}, f_3(x) = e^{-ix}$.

   (a) (4%) Prove that f_1, f_2, and f_3 are linearly independent.

   (b) (6%) Let $g_1(x) = 1$, $g_2(x) = cosx$, $g_3(x) = sinx$. Find an invertible 3×3 matrix P such that $g_j = \sum_{i=1}^{3} P_{i,j} f_i$.

2. (10%) Let V be the vector space of all functions from $\mathfrak{R}$ into $\mathfrak{R}$; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.

   (a) (4%) Prove that V_e and V_o are subspaces of V.

   (b) (3%) Prove that $V_e + V_o = V$.

   (c) (3%) Prove that $V_e \cap V_o = \{0\}$.

3. (10%) Let $S = \{v_1, v_2, \ldots, v_m\}$ be a set of vectors in $\mathfrak{R}^n$. Prove that if $m > n$, then S is linearly dependent.

4. (15%)

   (a) (8%) Let $V = M_3(\mathfrak{R})$ with inner product given by $\langle A, B\rangle = tr(AB')$, where $A, B \in M_3(\mathfrak{R})$ and B' is the transpose of B. Find an orthonormal basis for V relative to this inner product.

   (b) (7%) If V is as in problem 4. (a) and $W = \left\{\begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a, b, c \in \mathfrak{R}\right\}$, find $W^{\perp}$ and verify that $V = W \oplus W^{\perp}$.

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5. (10%) Let $T: M_2(\Re) \to M_2(\Re)$ be the linear operator defined by $T(X)=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}X+X\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$. Find the rank and nullity of T.

6. (10%) Let A be an $n\times n$ matrix. Prove that if B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.

7. (10%) Let V be the vector space of all functions from $\Re$ into $\Re$ which are continuous, i.e. the space of continuous real-valued functions on the real line. Let T be the linear operator on V defined by

$$(Tf)(x)=\int_0^x f(t)dt.$$ Prove that T has no *eigenvalue*.

8. (10%) Let $J: P_n \to P_{n+1}$ be the integration transformation defined by

$$J(P)=\int (a_0+a_1x+\ldots+a_nx^n)dx$$
$$=a_0x+\frac{a_1}{2}x^2+\ldots+\frac{a_n}{n+1}x^{n+1}$$

where $P=a_0+a_1x+\ldots+a_nx^n$. Find the matrix for T with respect to the standard bases for $\boldsymbol{P}_n$ and $\boldsymbol{P}_{n+1}$.

9. (15%) Suppose that we have a supply of 5000 units of S, 4000 units of T, and 2000 units of U, materials used in manufacturing products P and Q. We ask: If each unit of P uses 2 units of S, 0 units of T and 0 units of U, and each unit of Q uses 3 units of S, 4 units of T and 1 units of U, how many units p and q of P and Q should we make if we want to use up the entire supply?