The following problems may be answered in Chinese or English. You need to give all details in order to receive any point.

- P_2: all real polynomials of degree 2 or less
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- The set $\{0,1\}$ together with modulo-2 addition and multiplication, defined as $0+0=0$, $0+1=1$, $1+0=1$, $1+1=0$, $0\times 0=0$, $0\times 1=0$, $1\times 0=0$, $1\times 1=1$, is called a binary field and is denoted by $GF(2)$.

1. (15%) Prove that a transformation $T: R^n \to R^m$ is linear if and only if the following relationships hold for all vectors u an v in R^n and every scalar c.
   (a) $T(u+v) = T(u) + T(v)$　　(b) $T(cu) = cT(u)$

2. (10%) Show that if $f_1(x)$, $f_2(x)$, $g_1(x)$, $g_2(x)$ are differentiable functions, and if

$$W = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}, \text{ then } \frac{dW}{dx} = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}.$$

3. (20%) Construct the vector space V_5 of all 5-tuples over $GF(2)$. Find a three-dimensional subspace and determine its null space.

4. (15%) Given the matrices

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$ over $GF(2)$, show

   that the row space of G is the null space of H, and vice versa.

5. (10%) Find a weighted Euclidean inner product on R^n such that the vectors

$$v_1 = (1,0,0,\ldots,0),\ v_2 = (0,\sqrt{2},0,\ldots,0),\ v_3 = (0,0,\sqrt{3},0,\ldots,0) \ldots v_n = (0,0,0,\ldots,\sqrt{n})$$ form an orthonormal set.

6. (15%) **Prove that if $V_1, V_2, \ldots, V_k$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\{V_1, V_2, \ldots, V_k\}$ is a linearly independent set.**

7. (15%) **Let** $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$ **be the matrix of $T: P_2 \to P_2$ with respect to the basis $B=\{v_1,v_2,v_3\}$, where $v_1=3x+3x^2$, $v_2=-1+3x+2x^2$, $v_3=3+7x+2x^2$.**

   (a) (3%) **Find $[T(v_1)]_B$, $[T(v_2)]_B$, and $[T(v_3)]_B$.**
   (b) (3%) **Find $T(v_1)$, $T(v_2)$, $T(v_3)$.**
   (c) (7%) **Find a formula for $T(a_0+a_1x+a_2x^2)$.**
   (d) (2%) **Use the formula obtained in (c) to compute $T(1+x^2)$.**