科目：數學（以離散數學、線性代數為主）

編號：383 適用：資工系

考生注意：
1. 依次序作答，只要標明題號，不必抄題。
2. 答案必須寫在答案卷上，否則不予計分。
3. 限用藍、黑色筆作答；試題須隨卷繳回。

(以下各題均須寫出計算或證明過程方予計分)

1. In how many ways can 10 identical candy bars distributed among five children if
(5%) (a) there are no restrictions?
(5%) (b) each child gets at least one candy bar?

2. (10%) If 11 integers are selected from {1, 2, 3, …, 100}, prove that there are at least two, say x and y, such that $0 < \left|\sqrt{x} - \sqrt{y}\right| < 1$.

3. (10%) Determine how many integer solutions there are to $x_1 + x_2 + x_3 + x_4 = 19$, if $0 \le x_i < 8$ for all $1 \le i \le 4$.

4. (10%) Solve the recurrence relation $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$, $n \ge 0$, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

5. (5%) (a) A complete ternary (or 3-ary) tree $T = (V, E)$ has 34 internal vertices. How many edges does T have? How many leaves?
(5%) (b) How many internal vertices does a complete 5-ary tree with 817 leaves have?

6. (10%) Find a system of two linear equations in the variables x, y, and z whose solutions are given parametrically by $x = 3 + t$, $y = t$, and $z = 7 - 2t$.

7. (10%) Find the inverse of $A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$ by using row operations.

8. (15%) Given $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 8 \\ -2 & 4 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$.

(a) (5%) Find an orthonormal basis for the column space of A.

(b) (10%) Find the orthogonal projection of $\mathbf{b}$ on the column space of A, and find the least squares solutions of the linear system $A\mathbf{x} = \mathbf{b}$.

9. (5%) Find a weighted Euclidean inner product on $\boldsymbol{R}^n$ such that the vectors $v_1 = (1, 0, 0, \ldots, 0)$, $v_2 = (0, \sqrt{2}, 0, \ldots, 0)$, $v_3 = (0, 0, \sqrt{3}, 0, \ldots, 0)$, …, $v_n = (0, 0, 0, \ldots, \sqrt{n})$ form an orthonormal set.

10. (10%) Let $s_1(t)$ and $s_2(t)$ be two real-valued functions shown in Figure 1.

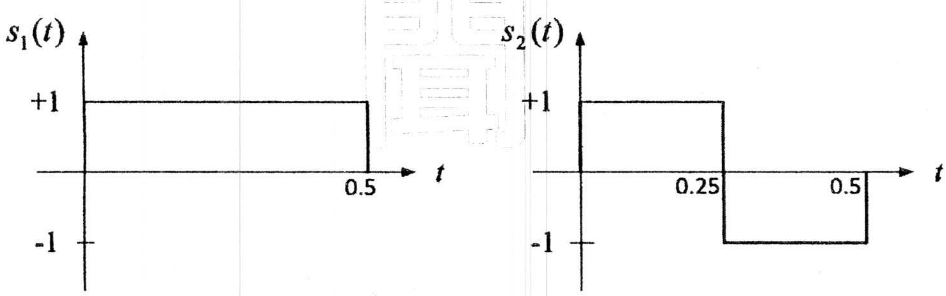

Figure 1. $s_1(t)$ and $s_2(t)$, where $0 \le t \le 0.5$.

(a) (2%) Let $S = \{s_1(t), s_2(t)\}$, show that S is a linear independent set.

(b) (3%) Suppose that V is the real vector space spanned by $s_1(t)$ and $s_2(t)$. Let V have the inner product $<p(t), q(t)> \triangleq \int_0^{0.5} p(t)q(t)dt$, where $p(t), q(t) \in V$; and the *norm* (or *length*) of $p(t)$ is defined by $\|p(t)\| = \sqrt{<p(t), p(t)>}$. Show that $<s_1(t), s_2(t)> = 0$ and find an orthonormal basis for V.

(c) (2%) Let $r(t) = 3\, s_1(t) + 2\, s_2(t)$, plot $r(t)$.

(d) (3%) Find $< r(t), s_2(t)>$.